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ON AN ALTERNATIVE IN A DIFFERENTIAL EVASION GAME WITH INCOMPLETE INFORMATION

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A conflict game situation is analyzed, made up of the problems of evading a closed target set and of contact with this set. It is assumed that the player resolving the evasion problem has incomplete information on the system's phase coordinates. It is shown that either the evasion problem or the encounter problem is always solvable in the game situation mentioned. The arguments are based on the extremal construction from [1, 2], suitably modified in [3 — 5] to allow for the specific nature of control problems under incomplete information (*).

1. Let the motion of a controlled system be described by the vector differential equation

$$dx / dt = A(t) x + f_*(t, u, v), \quad u \in P(t), \quad v \in Q(t) \qquad (1.1)$$

Here x is a vector in an n-dimensional Euclidean space R^n, and, u and v are control vectors; $P(t)$ and $Q(t)$ are compact sets in finite-dimensional normed spaces, bounded in every finite interval and measurably dependent on t. The matrix $A(t)$ is assumed Lebesgue summable in every finite interval, while the function $f(t, u, v)$ is continuous in all the arguments. A closed set M_* is delineated in the space $R^1 \times R^n$ of the variables t and x. Treating v as a control vector and u as some poorly predictable noise, we consider the problem of the phase state $x[t]$ of system (1.1) evading the set M_* till some specified instant ϑ. We assume the vector $x \in R^n$ is subject to a nonsingular linear transformation (for instance, see [3]), as a result of which Eq. (1.1) becomes

$$dx / dt = f(t, u, v), \quad u \in P(t), \quad v \in Q(t) \qquad (1.2)$$

the function $f(t, u, v)$ remains continuous in all arguments and the set M_* turns into a new set M, also remaining closed.

The control process is complicated by the absence of complete information on the system's phase states realized and proceeds by the following discrete approximate scheme. At an instant τ_i the controller knows a certain compact set $G[\tau_i] \subset R^n$ containing the state $x[\tau_i]$ of system (1.2), realized by this instant. On the basis of this information

*) Kriazhimskii, A. V., A linear guidance game with incomplete information. Sverdlovsk, Preprint, UNTs Akad. Nauk SSSR, 1975.

he chooses, at the instant τ_i, his Lebesgue measurable control $v[t]$ for the next interval $[\tau_i, \tau_{i+1})$. At the instant τ_{i+1} he knows a new information set $G[\tau_{i+1}]$ and the process is repeated. In the control process he can meet up with any realization $u[t]$ of noise u, Lebesgue measurable and satisfying the inclusion $u[t] \in P(t)$, and with any realizations of the information sets $G[\tau_i]$ containing the current states $x[\tau_i]$ of system (1.2) and satisfying the conditions

$$G[\tau_i] \subset G\{G[\tau_{i-1}], \tau_{i-1}, \tau_i\}, \quad G[\tau_i] \in \Gamma(\tau_i) \tag{1.3}$$

Here $G\{G[\tau_{i-1}], \tau_{i-1}, \tau_i\}$ denotes the set of those states into which the vector $x[t]$ can be led by the instant τ_i under law (1.2), starting the motion at instant τ_{i-1} from the points of set $G[\tau_{i-1}]$ under the action of the controls $v[t]$ and perturbations $u[t]$ realized in the interval $[\tau_{i-1}, \tau_i)$; $\Gamma(t)$ denotes some t-dependent family of sets in R^n, whose properties are defined more exactly below. Condition (1.3) assumes that at the instant τ_i the controller receives information on the control $v[t]$ and perturbation $u[t]$ realized $(t \in [\tau_{i-1}, \tau_i))$; he can use this information only to update the information set at the instant τ_i, but he cannot use it when forming his control $v[t]$. The second of conditions (1.3), constraining the class of information sets, characterizes the quality of the observation method being used.

Having adopted the control scheme mentioned, it is natural to replace the original problem with incomplete information by the problem of controlling the information set; it is also natural to replace the evasion condition for the not exactly known state $x[t]$ by the condition

$$\{t, G[t]\} \cap M = \varnothing, \qquad t \in [t_0, \vartheta] \tag{1.4}$$

In order to evaluate the results ensured by some control law or other, it is convenient to imbed the problem being analyzed into the plan of some antagonistic game set up between the player, namely, the controller choosing $v[t]$, and a fictitious player who has at his disposal the realizations of perturbations $u[t]$ and of the information sets $G[t]$ within the limits of constraints (1.3). It turns out to be convenient for the fictitious player — the opponent — (when viewing the problem from his "point of view") to retain the "positional" nature of the formation of the perturbations $u[t]$ and of the information sets $G[t]$. Namely, these players form the objects mentioned at discrete instants τ_i, which take place by the following hierarchical scheme: the set $G\{G[\tau_{i-1}], \tau_{i-1}, \tau_i\}$ prescribes the set $G[\tau_i]$ satisfying conditions (1.3); the control $v[t]$, which will act on the system during the interval $[\tau_i, \tau_{i+1})$, is formed from the set $G[\tau_i]$, and the process is repeated.

To preserve terminology, the player, i.e. the controller who has the control $v[t]$ at his disposal and who wishes to effect the evasion (1.4), is named the second player, while his opponent, who forms the perturbations $u[t]$ and the information sets, is named the first player. Since the game being analyzed is antagonistic, it is natural to present the condition for terminating the process, successfully for the first player, in terms of the information sets as

$$\{\tau, G[\tau]\} \cap M \neq \varnothing \tag{1.5}$$

for some $\tau \in [t_0, \vartheta]$.

2. The mathematical formalizations of the conflict control problems mentioned is based on the substitution described above of the original problems with incomplete information by problems of controlling an information domain treated as a new generalized phase space comp (R^n) formed by all possible compact sets from R^n with the Hausdorff metric. Let a nonempty closed set $\Gamma(t) \subset$ comp (R^n) be associated with every $t \in R^1$ and let the following be satisfied:

Condition 2.1. $G + y \in \Gamma(t)$ for all $G \in \Gamma(t)$, $y \in R^n$, $t \in R^1$. The symbol $A + B$ denotes the algebraic sum of sets $A, B \subset R^n$; if one of them, say B, consists of a single vector $b \in R^n$ (i.e., $B = \{b\}$), then instead of writing $A + \{b\}$ we simply use the notation $A+b$.

The sets $\Gamma(t)$, characterizing the quality of the method being used to observe system (1.1), can be treated in the ensuing presentation as distinctive constraints on the "information resources" of the first player prescribing the set $G[t]$. The pairs $\{t, G\} \in R^1 \times$ comp (R^n) are called positions. In accord with the above mentioned principle for forming the controls $v[t]$ by the second player, i.e. the controller, the control laws used by him, i.e. the strategy V, are identified with mappings which associate with every position $\{t, G\}$ a function $v[\tau]$, $(\tau \in [t, \infty))$, measurable dependent on τ and satisfying the condition $v[\tau] \in Q(\tau)$. The laws for forming the information domains $G[t]$ and perturbations $u[t]$ by the first player, viz., his strategies J and U, are identified with mappings which associate with every position $\{t, G\}$ a set $G' = J(t, G)$ satisfying the conditions

$$G \subset G, \quad G \in \Gamma(t)$$

and a measurable function $u[\tau]$, $(\tau \in [t, \infty))$ satisfying the inclusion $u[\tau] \in P(\tau)$, respectively.

Let $\Delta = \{\tau_i : \tau_0 = t_0, \tau_{i+1} > \tau_i, i = 0, 1 \ldots\}$ be some partitioning of the semi-axis $[t_0, \infty)$ by the intervals $[\tau_i, \tau_{i+1})$, $u[t]$ be some measurable function with values in set $P(t)$, and V be some strategy. Every function $G_\Delta[t] = G_\Delta[t, t_0, G_0, V, u[\cdot]]$ with values in the space comp (R^n), defined by the following recurrence relations

$$G_\Delta[t] = G[\tau_i] + \int_{\tau_i}^{t} f(\tau, u[\tau], v_i[\tau])\, d\tau, \quad t \in [\tau_i, \tau_{i+1})$$

$$G_\Delta[\tau_{i+1}] \subset G[\tau_i] + \int_{\tau_i}^{\tau_{i+1}} f(\tau, u[\tau], v_i[\tau])\, d\tau$$

$$G_\Delta[t_0] = G_0, \quad G[\tau_{i+1}] \in \Gamma(\tau_{i+1}), \quad v_i[\cdot] = V(\tau_i, G_\Delta[\tau_i])$$

is called an approximate motion from position $\{t_0, G_0\}$, corresponding to the elements chosen. Analogously, every function $G_\Delta[t] = G_\Delta[t, t_0, G_0, U, J, v[\cdot]]$ with values in the space comp (R^n), defined by the recurrence relations

$$G_\Delta[t] = G[\tau_i] + \int_{\tau_i}^{t} f(\tau, u_i[\tau], v[\tau])\, d\tau, \quad t \in [\tau_i, \tau_{i+1})$$

$$G_\Delta[\tau_{i+1}] = J(\tau_{i+1}, G_\Delta[\varphi_i] + \int_{\tau_i}^{\tau_{i+1}} f(\tau, u_i[\tau], v[\tau])\, d\tau$$

$$G_\Delta^-[t_0] = G_0, \quad u_i[\cdot] = U(\tau_i, G_\Delta[\tau_i])$$

is called an approximate motion from position $\{t_0, G_0\}$, corresponding to strategies U and J, to partitioning $\Delta = \{\tau_i\}$, and to some measurable function $v[t]$ satisfying the condition $v[t] \in Q(t)$.

In what follows it is convenient to pass from approximate motions to certain ideal motions, as was done in [1, 7] for game problems with complete information. Let $G: R^1 \to \mathrm{comp}(R^n)$ be some mapping. By $\mathrm{gr}_{[t_1,t_2]}G$ we denote a graph of mapping G, considered in the interval $[t_1, t_2]$ and defined by the relation $\mathrm{gr}_{[t_1,t_2]} G = [\{t, x\} \in [t_1, t_2] \times R^n : x \in G(t)]$. Every function $G[t]$ with values in the space $\mathrm{comp}(R^n)$, for which we can find a sequence of approximate motions $G_{\Delta(k)}[t] = G_{\Delta(k)}[t, t_0, G_0^{(k)}, V, u^{(k)}[\cdot]]$ in every interval $[t_0, \vartheta]$, such that

$$\mathrm{cl}(\mathrm{gr}_{[t_0,\vartheta]}G_{\Delta(k)}) \to \mathrm{gr}_{[t_0,\vartheta]}G = \mathrm{cl}(\mathrm{gr}_{[t_0,\vartheta]}G), \quad G_0^{(k)} \to G_0 \tag{2.1}$$
$$\sup_i(\tau_{i+1}^{(k)} - \tau_i^{(k)}) \to 0$$

as $k \to \infty$, is called a motion $G[t, t_0, G_0, V]$ from position $\{t_0, G_0\}$, generated by strategy V. We remark that the symbol $\mathrm{cl}(X)$ denotes the closure of a set X in the appropriate space and that convergence in (2.1) is to be understood in the sense of the Hausdorff metric in the appropriate space. The motions generated by strategies U and J are defined similarly. Namely, every function $G[t]$ with values in the space $\mathrm{comp}(R^n)$, for which we can find a sequence of approximate motions $G_{\Delta(k)}[t] = G_{\Delta(k)}[t, t_0, G_0, U, J, v^{(k)}[\cdot]]$, in every interval $[t_0, \vartheta]$, such that

$$\mathrm{cl}(\mathrm{gr}_{[t_0,\vartheta]}G_{\Delta(k)}) \to \mathrm{gr}_{[t_0,\vartheta]}G = \mathrm{cl}(\mathrm{gr}_{[t_0,\vartheta]}G)$$
$$\sup_i(\tau_{i+1}^{(k)} - \tau_i^{(k)}) \to 0$$

as $k \to \infty$, is called a motion $G[t, t_0, G_0, U, J]$ from position $\{t_0, G_0\}$, corresponding to strategies U and J.

The sets of all possible motions generated by strategies V or by U and J from the position $\{t_0, G_0\}$ are denoted $\Pi(t_0, G_0, V)$ and $\Pi(t_0, G_0, U, J)$, respectively. The following is valid:

Lemma 2.1. For any strategies V, U, J the sets $\Pi(t_0, G_0, V)$ and $\Pi(t_0, G_0, U, J)$ contain at least one common element. All elements of the sets mentioned are functions upper-semicontinuous with respect to inclusion [5], with values in the space $\mathrm{comp}(R^n)$ and satisfying the conditions

$$G[t_0] = G_0, \quad G[t] \neq \varnothing, \quad t \in [t_0, \infty) \tag{2.2}$$

Just as in control problems with complete information [1, 5], the given abstract definition of strategies and motions admits of a natural transition, at least for the second player — the controller — to control procedures realizable in practice. This transition is effected by reverting to the approximate motions and to the discrete scheme for forming the controls; all the statements made for motions $G[t]$ have their approximate analogies.

We turn now to a formalized posing of the problem. The problem of the second player — the controller — can be posed as follows:

Problem 2.1. For a position $\{t_0, G_0\}$ find an open neighborhood $H(M)$ of the set M and construct a strategy V° such that the condition

$$\{t, G[t]\} \cap H(M) = \varnothing, \ t \in [t_0, \infty) \tag{2.3}$$

is satisfied for every motion $G[t] = G[t, t_0, G_0, V^\circ]$

The first player's counter-problem, in the concepts formalized above of strategies U

and J and of the motions corresponding to them, can be posed as follows:

Problem 2.2. For a position $\{t_0, G_0\}$ construct strategies U° and J° which would ensure the fulfillment of the condition

$$\{\tau, G[\tau]\} \cap M \neq \varnothing \tag{2.4}$$

at some instant $\tau \in [t_0, \vartheta]$ for every motion $G[t] = G[t, t_0, G_0, U^\circ, J^\circ]$

We note that the condition $\Pi(t_0, G_0, V) \cap \Pi(t_0, G_0, U, J) \neq \varnothing$, valid by virtue of Lemma 2.1 for all possible positions $\{t_0, G_0\}$ and strategies V, U, J, enables us to unite Problems 2.1 and 2.2 and to call their union a game.

3. The approach to the problems being analyzed is based on the extremal construction from [1], modified as in [3 — 5] to the distinctive features of the given game problems. Let us describe the fundamental elements of the extremal construction mentioned. First of all we consider the case where a saddle point exists in the small game [5], i.e., we assume that the condition

$$\min_{u \in P(t)} \max_{v \in Q(t)} sf(t, u, v) = \max_{v \in Q(t)} \min_{u \in P(t)} sf(t, u, v) \tag{3.1}$$

is satisfied for all $t \in R^1$ and $s \in R^n$.

Let $W \subset R^1 \times \operatorname{comp}(R^n)$ be some set. The following concept, named stability, is of subsequent importance. For arbitrary measurable functions $u(t)$ and $v(t)$ satisfying the conditions $u(t) \in P(t)$ and $v(t) \in Q(t)$, $(t \in [t_*, \infty))$, we set

$$F_u(t, v(t)) = \operatorname{co}\{f : f = f(t, u, v(t)), u \in P(t)\} \tag{3.2}$$

$$F_v(t, u(t)) = \operatorname{co}\{f : f = f(t, u(t), v), v \in Q(t)\} \tag{3.3}$$

and we consider the differential inclusions

$$dx / dt \in F_v(t, u(t)), \quad x(t_*) = 0 \tag{3.4}$$

$$dx / dt \in F_u(t, v(t)), \quad x(t_*) = 0 \tag{3.5}$$

The set W is said to be v-stable if for every choice of position $\{t_*, G_*\} \in W$ of instant $t^* > t_*$, of measurable function $u(t) \in P(t)$ $(t \geqslant t_*)$, and of set $G^* \subset G_*$, $G^* \in \Gamma(t^*)$, we can find an obsolutely continuous solution $x(t)$ of inclusion (3.4) such that $\{t^*, G^* + x(t^*)\} \in W$. However, if it happens that for every choice of position $\{t_*, G_*\} \in W$, of instant $t^* > t_*$, and of measurable function $v(t)$ satisfying the condition $v(t) \in Q(t)$ $(t \geqslant t_*)$ we can find an absolutely continuous solution $x(t)$ of inclusion (3.5) and a set $G^* \subset G_*$, $G^* \in \Gamma(t^*)$, such that either $\{\tau, G_* + x(\tau)\} \cap M \neq \varnothing$ for some $\tau \in [t_*, t^*]$ or $\{t^*, G^* + x(t^*)\} \in W$, then set W is said to be $u - j$-stable (relative to target M).

We now turn to a description of another important element of the construction being considered, viz., extremal strategies. Let us consider a certain closed set W in the space $R^1 \times \operatorname{comp}(R^n)$ with the metric

$$\rho(\{t_1, G_1\}, \{t_2, G_2\}) = \max \{|t_1 - t_2|, \operatorname{dist}(G_1, G_2)\} \tag{3.6}$$

Let $\{t_*, G_*\}$ be an arbitrary position. By $S(t_*, G_*)$ we denote the collection of all possible vectors $s \in R^n$ satisfying the conditions

$$\{t_*, G_* + s\} \in W, \quad \|s\| = \min [\|s'\|, \ s' \in R^n : \{t_*, G_* + s'\} \in W] \tag{3.7}$$

We construct a strategy $V^{(e)}$ which, by analogy with [1, 5], we say is *extremal to set* W. When $S(t, G) \neq \varnothing$ we choose an arbitrary vector $s_* \in S(t, G)$ and we define the sets $Q^{(e)}(\tau)$ $(\tau \in [t, \infty))$ made up of all possible vectors $v_* \in Q(\tau)$ satisfying the condition

$$\min_{u \in P(\tau)} s_* f(\tau, u, v_*) = \max_{v \in Q(\tau)} \min_{u \in P(\tau)} s_* f(\tau, u, v) \quad (3.8)$$

Using the measurability of the many-valued mappings of $P(t)$ and $Q(t)$, it can be shown that the sets $Q^{(e)}(\tau)$ also depend measurably on parameter τ. For the position $\{t, G\}$ being considered we now set $V^{(e)}(t, G) = v[\cdot]$, where $v[\tau]$ is an arbitrary measurable function satisfying the condition $v[\tau] \in Q^{(e)}(\tau)$, $(\tau \in [t, \infty))$. However, if the set $S(t, G)$ turns out to be empty for a position $\{t, G\}$, then we set $V^{(e)}(t, G) = v[\cdot]$, where $v[\tau]$ is an arbitrary measurable function satisfying the condition $v[\tau] \in Q(\tau)$, $(\tau \in [t, \infty))$.

The strategy $U^{(e)}$ extremal to W is defined analogously. Namely, if $S(t, G) \neq \varnothing$, we should choose an arbitrary vector $s_* \in S(t, G)$ and construct the sets $P^{(e)}(\tau)$, $(\tau \in [t, \infty))$ made up of all possible vectors $u_* \in P(\tau)$ satisfying the condition

$$\max_{v \in P(\tau)} - s_* f(\tau, u_*, v) = \min_{u \in P(\tau)} \max_{v \in Q(\tau)} - s_* f(\tau, u, v) \quad (3.9)$$

Further, for the position chosen we should set $U^{(e)}(t, G) = u[\cdot]$, where $u[\tau]$ is an arbitrary measurable function satisfying the condition $u[\tau] \in P^{(e)}(\tau)$ $(\tau \in [t, \infty))$. However, if $S(t, G) = \varnothing$, we should set $U^{(e)}(t, G) = u[\cdot]$, where $u[\tau]$ is an arbitrary measurable function satisfying the condition $u[\tau] \in P(\tau)$ $(\tau \in [t, \infty))$.

The strategy $J^{(e)}$ extremal to set W takes a somewhat different form in the case at hand. For an arbitrary position $\{t, G\}$ we denote by $\Xi(t, G)$ the collection of all possible sets $G' \in \mathbf{comp}(R^n)$ such that

$$\{t, G' + y'\} \in W, \quad G' \subset G, \quad G' \in \Gamma(t) \quad (3.10)$$

for various $y' \in R^n$ and

$$\| y' \| \leqslant \| y \| \quad (3.11)$$

for every vector $y \in R^n$ for which we can find a set $G'' \subset G$, $G'' \in \Gamma(t)$ such that $\{t, G'' + y\} \in W$. Thus, the set $\Xi(t, G)$ is a collection of all compacta $G' \subset G$, $G' \in \Gamma(t)$, for which the inclusion $\{t, G' + y'\} \in W$ is ensured by a shift in the minimal norm by a vector $y' \in R^n$. Strategy $J^{(e)}$ is now defined as follows. When $\Xi(t, G) \neq \varnothing$ we set $J^{(e)}(t, G) = G'$, where G' is an arbitrary element from the set $\Xi(t, G)$. However, if the set $\Xi(t, G)$ is empty for the position chosen, then the set $J^{(e)}(t, G) = G'$, where G' is an arbitrary set satisfying the conditions $G' \subset G$, $G' \in \Gamma(t)$.

Let the symbol $G^{[\varepsilon]}$ denote a closed ε-neighborhood of set $G \subset R^n$. The role of the extremal strategies is clarified in the following statements.

Lemma 3.1. If a number $\alpha_0 > 0$ exists for a position $\{t_0, G_0\}$, such that $\{t_0, G_0^{[\alpha_0]}\} \in W$ and if a closed set $W \subset R^1 \times \mathbf{comp}(R^n)$ is v-stable and is such that the validity of the inclusion $\{t, G\} \in W$ implies the validity of the inclusion $\{t, G'\} \in W$ for all $G' \in \mathbf{comp}(R^n)$, $G' \subset G$, then the strategy $V^{(e)}$, extremal to set W for all motions $G[t] = G[t, t_0, G_0, V^{(e)}]$, ensures the satisfaction of the condition

$$\{t, G[t]\} \in W, \quad t \geqslant t_0 \quad (3.12)$$

Lemma 3.2. If $\{t_0, G_0\} \in W$ and the closed set $W \subset R^1 \times \text{comp}(R^n)$ is $u - j$-stable, then the strategies $U^{(e)}$ and $J^{(e)}$, extremal to the given set W, ensure the satisfaction of the condition

$$\{t, G[t]\} \in W, \quad t \in [t_0, \tau) \tag{3.13}$$

for every motion $G[t] = G[t, t_0, G_0, U^{(e)}, J^{(e)}]$ for which

$$\{t, G[t]\} \cap M = \varnothing \tag{3.14}$$

for all $t \in [t_0, \tau]$.

The statements made can be proved along the lines used, for instance, in [5] for an ordinary differential game with complete information and we don't do so here. It should be noted that Lemmas 3.1 and 3.2 remain valid in the approximate version as well. The motions $G[t]$ should be replaced by their approximate analogs, viz., by the motions $G_\Delta[t]$, while the sets W and M, by open ε-neighborhoods of them in the appropriate spaces. In this case the statements are valid for all approximate motions $G_\Delta[t]$ with partitionings $\Delta = \{\tau_i\}$ whose diameters $d(\Delta) = \sup_i (\tau_{i+1} - \tau_i)$ do not exceed some fairly small number $\delta = \delta_\varepsilon$.

4. Relying on Lemmas 3.1 and 3.2 and assuming the satisfaction of

Condition 4.1. $\Gamma(t^*) \subset \Gamma(t_*)$ for all $t^* > t_*$, we prove the validity of the following alternative statement which is the main result of the present paper.

Theorem 4.1. One and only one of the following two assertions is valid for any number ϑ and position $\{t_0, G_0\}$: either (I) Problem 2.1 of evasion up to the instant ϑ is solvable for the position $\{t_0, G_0\}$ or (II) Problem 2.2 on contact by the instant ϑ is solvable for the position $\{t_0, G_0\}$.

Let us sketch the proof of Theorem 4.1, following the sequence of reasonings from [5]. Suppose that some ϑ has been chosen. By $W^{(\vartheta)}$ we denote the set of all possible positions $\{t, G\}$ for each of which, taken as the initial position, Problem 2.1 of evasion from set M up to instant ϑ is not solvable. Analogously to [5] it can be shown that the set $W^{(\vartheta)}$ indicated is closed. However, it happens to be singularly important that set $W^{(\vartheta)}$ is $u - j$-stable. To prove this we accept to the contrary that it is not. Then we can find a position $\{t_*, G_*\} \in W^{(\vartheta)}$, an instant $t^* > t_*$, and a measurable function $v_*(t)$ satisfying the condition $v_*(t) \in Q(t)$ $(t \in [t_*, t^*])$ such that for every absolutely continuous solution $x(t)$ of the inclusion

$$dx / dt \in F_u(t, v_*(t)), \quad x(t_*) = 0 \tag{4.1}$$

the following conditions are satisfied:

$$\{\tau, G_* + x(\tau)\} \cap M = \varnothing \tag{4.2}$$

for all $\tau \in [t_*, t^*]$

$$\{t^*, G + x(t^*)\} \notin W \tag{4.3}$$

for all $G \subset G_*$, $G \in \Gamma(t^*)$. Note that because the set of solutions of differential inclusion (4.1) is closed in the uniform metric and because set M is closed, condition (4.2) remains valid also for some sufficiently small open neighborhood $M^{(\varepsilon_0)}$ of set M.

Let $\Sigma^* \subset \text{comp}(R^n)$ be the set of all possible elements of G' of the form

$$G' = G + x(t^*), \quad G \subset G_*, \quad G \in \Gamma(t^*) \tag{4.4}$$

where $x(\cdot)$ is some solution of inclusion (4.1). Using the definition of sets $W^{(\vartheta)}$ and Σ^* and the Condition 4.1 and arguing just as in the analogous situation in the ordinary position differential game [5], it can be shown that for every element $G^* \in \Sigma^*$ we can find a strategy V^*, an open neighborhood $H^*(M)$ of set M, and a number $\delta^* > 0$ such that for all motions $G[t] = G[t, t^*, G, V^*]$ the satisfaction of the condition

$$\{t, G[t]\} \cap H^*(M) = 0, \quad t \in [t^*, \vartheta] \tag{4.5}$$

is ensured if only the element $G \in \text{comp}(R^n)$ satisfies the inequality

$$\text{dist}(G, G^*) \leqslant \delta^* \tag{4.6}$$

Here dist (G, G^*) is the Hausdorff distance between sets G and G^*. Taking into account the given property of the elements $G^* \in \Sigma^*$ and also the compactness of set Σ^*, it is easy to be convinced that we can find a finite system of sets

$$\sigma_i = \{G \in \text{comp}(R^n) : \text{dist}(G, G_i^*) \leqslant \delta_i^*\}, \quad G_i^* \in \Sigma^*$$

covering set Σ^* and the strategies V_i^* ($i \in \{1, 2, \ldots, m\}$) such that for every element $G \in \sigma_i$ the strategy V_i^* ensures the evasion

$$\{t, G[t]\} \cap H_i^*(M) = \varnothing, \quad t \in [t^*, \vartheta] \tag{4.7}$$

of all motions $G[t] = G[t, t^*, G, V_i^*]$ from some open neighborhood $H_i^*(M)$ of set M in the interval $[t^*, \vartheta]$.

Let ε^* be the minimum one of the numbers $\varepsilon_0/2, \delta_1^*/2, \delta_2^*/2, \ldots, \delta_m^*/2$, and let $\Lambda^*(G) = \{G' : G' \subset G, G' \in \Gamma(t^*)\}$. Arguing by contradiction, it is easy to see that a number $\alpha_0 > 0$ exists such that

$$\Lambda^*(G_*^{[\alpha_0]}) \subset (\Lambda^*(G_*))^{(\varepsilon^*)} \tag{4.8}$$

Let $W_*^{(\vartheta)}$ denote the closure of the set consisting of all possible positions $\{t, G\}$ such that either $t \in [t_*, t^*]$ and

$$G = G' + x(t^*), \quad G' \in \text{comp}(R^n), \quad G' \subset G_*^{[\alpha_0]} \tag{4.9}$$

where $x(\cdot)$ is some solution of inclusion (4.1), or $t \in (t^*, \infty)$ and

$$G = G[t, t^*, G', V_j^*] \tag{4.10}$$

for some $j \in \{1, 2, \ldots, m\}$ and some motion $G[\cdot, t^*, G', V_j^*]$ and element $G' \in \sigma_j$.

Using condition (4.1) it is easy to see that the set $W_*^{(\vartheta)}$ constructed is v-stable and that the condition

$$\{t, G\} \cap H_*(M) = \varnothing \tag{4.11}$$

is satisfied for every position $\{t, G\} \in W_*^{(\vartheta)}$, $t \in [t_*, \vartheta]$. Here $H_*(M)$ is some open neighborhood of set M, as which we can select, for example, the intersection of the neighborhoods $H_0(M) = M^{(\varepsilon_0/2)}, H_1^*(M), H_2^*(M), \ldots, H_m^*(M)$.

According to the definition of set $W_*^{(\vartheta)}$ the inclusion $\{t_*, G_*^{[\alpha_0]}\} \in W_*^{(\vartheta)}$ is valid for the position $\{t_*, G_*\}$; whence, with due regard to the v-stability of set $W_*^{(\vartheta)}$, in accord with Lemma 3.1 we conclude that the strategy $V^{(e)}$ extremal to the given set ensures the evasion

$$\{t, G[t]\} \cap H_*(M) = \varnothing, \quad t \in [t_*, \vartheta] \tag{4.12}$$

of all motions $G[t] = G[t, t_*, G_*, V^{(e)}]$ from an open neighborhood $H_*(M)$ of set M in the interval $[t_*, \vartheta]$, and so, Problem 2.1 of evasion is solvable for the position $\{t_*, G_*\}$. However, the latter contradicts the choice of the position $\{t_*, G_*\} \in W^{(\vartheta)}$ as a position for which Problem 2.1 is not solvable; by the same token the u - j-stability of set $W^{(\vartheta)}$ is proved.

Now we take into account the condition

$$\{\vartheta, G\} \cap M \neq \varnothing, \quad \{\vartheta, G\} \in W^{(\vartheta)} \tag{4.13}$$

following directly from the definition of set $W^{(\vartheta)}$ and, by virtue of Lemma 3.2, we conclude that Problem 2.2 on encounter with set M by the instant ϑ is solvable for every position $\{t_0, G_0\} \in W^{(\vartheta)}$; as the strategies resolving this problem we can take the strategies $L^{(e)}$ and $J^{(e)}$ extremal to set $W^{(\vartheta)}$. Thus, for every position $\{t_0, G_0\}$, Problem 2.1

is solvable if only $\{t_0, G_0\} \in W^{(\vartheta)}$ and Problem 2.2 is solvable if $\{t_0, G_0\} \notin W^{(\vartheta)}$, and that, with due regard to Lemma 2.1, guarantees the impossibility of the simultaneous solvability of these problems and proves Theorem 4.1.

The approximate analogs of Lemmas 3.1 and 3.2 enable us to obtain the following approximate version of Theorem 4.1.

Theorem 4.2. Suppose that some number ϑ has been chosen and that some position $\{t_0, G_0\}$ has been fixed. Then, if the first assertion of Theorem 4.1, i.e. that Problem 2.1 is solvable, is correct, then we can find a strategy V°, a number $\delta_0 > 0$ and some open neighborhood $H_0(M)$ of set M such that the satisfaction of the condition

$$\{t, G_\Delta[t]\} \cap H_0(M) = \varnothing, \quad t \in [t_0, \vartheta] \tag{4.14}$$

is ensured for all approximate motions $G_\Delta[t] = G_\Delta[t, t_0, G_0, V^\circ, u[\cdot]]$ with the step $d(\Delta) = \sup_i (\tau_{i+1} - \tau_i)$ of the partitioning $\Delta = \{\tau_i\}$, satisfying the condition $d(\Delta) < \delta_0$. To the contrary, if the second assertion of Theorem 4.1, i.e. that Problem 2.2 is solvable, is correct, then for every choice of number $\varepsilon > 0$ we can find a number $\delta_\varepsilon > 0$ such that certain strategies U° and J° ensure the satisfaction of the condition

$$\{\tau, G_\Delta[\tau]\} \cap M^{[\varepsilon]} \neq \varnothing, \qquad \tau \in [t_0, \vartheta] \tag{4.15}$$

for all approximate motions $G_\Delta[t] = G_\Delta[t, t_0, G_0, U^\circ, J^\circ; v[\cdot]]$ with the step $d(\Delta) = \sup_i (\tau_{i+1} - \tau_i)$ of partitioning Δ not exceeding δ_ε.

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